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An accurate method for solving the Orr-Sommerfeld equation

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SUMMARY

The method consists in integrating the Orr-Sommerfeld equation in the direction from the free stream toward the wall. In order to satisfy the boundary conditions at the wall, two linearly independent solutions have to be found. To prevent numerical solutions from becoming linearly dependent, the method of order reduction instead of repeated orthogonalization has been used. The method has been applied to calculate the neutral curve for the Blasius profile.

1. Introduction

The Orr-Sommerfeld equation governs the stability of a two-dimensional incompressible laminar parallel flow with respect to infinitesimal two-dimensional disturbances. In the past a number of methods has been presented to calculate eigenvalues and eigenfunctions of the Orr-Sommerfeld equation. Osborne [1] and Jordinson [2] replaced the differential equation by a set of difference equations which leads to a matrix eigenvalue problem. This allows to calculate several eigenvalues but the accuracy is limited. Orszag [3] used expansions in the Chebyshev polynomials and the QR matrix eigenvalue algorithm. Mack [4] and Monkewitz [5] integrate the two decreasing solutions of the Orr-Sommerfeld equation from some large value of the independent variable y to the wall $y = 0$. In order to prevent that the two solutions, which increase in the direction of integration, become linearly dependent, Mack uses a Gram-Schmidt orthonormalization procedure and Monkewitz a pseudo-orthogonalization method. However, these have to be applied several times during the integration process.

In the present paper a method is presented which also integrates the differential equation from a large value of y to $y = 0$ but which avoids the repeated orthogonalization. It is based upon the method of order reduction, see [6], which originally is due to d'Alembert. If φ_1 is the solution which increases most strongly in the direction of integration, a second solution is obtained by putting $\varphi = \psi\varphi_1$ which leads to a differential equation for ψ of an order, one lower than that of the original equation for ψ. Application of the method of order reduction to a (different) numerical problem has been performed by Should [7]. This was brought to the attention of the authors by Prof. J. H. Ferziger from Stanford University.

Although the Orr-Sommerfeld equation is derived for parallel flow, it has frequently been applied for investigating the stability of the Blasius boundary layer profile.This will also be done in the present paper, but it is intended to consider later the influence of the non-parallellity of the flow (see also Barry and Ross [8] and Saric and Nayfeh [9]).

Results are presented for the neutral stability curve in the $\tilde{R},\tilde{\alpha}$-plane, where $\tilde{R}$ is the Reynolds number and $\tilde{\alpha}$ the wave number, both made dimensionless by aid of the displacement thickness δ_1^* of the boundary layer. The lowest Reynolds number at which the Blasius boundary layer becomes unstable is 519.060. The method can also be used to calculate for given $\tilde{R}$ and $\tilde{\alpha}$ the damping of the oscillation. This has been performed for a standard case which also was considered in [2], [4] and by Grosch and Salwen [10]. Results are in excellent agreement.

2. The Orr-Sommerfeld equation

A flat plate is assumed to lie in the plane $y = 0$ with its leading edge along the z-axis. The plate is placed in a uniform stream of velocity U_∞^*, parallel to the plate and in the direction of the x-axis. x, y, z is a Cartesian system of axes. Physical quantities are made dimensionless by aid of the free stream velocity U_∞^* and the length

$$L^* = \sqrt{\frac{\nu^* x^*}{U_\infty^*}},$$

where ν^* is the kinematic viscosity. Asterisks refer to dimensional quantities. The Reynolds number is defined as

$$R = \frac{U_\infty^* \, L^*}{\nu^*}.$$

The displacement thickness becomes $\delta_1^* = \beta L^*$, where $\beta = 1.7207876573$. The perturbation stream function is assumed to be given by the real part of the expression.

$$\Psi(x, y, t) = \varphi(y) e^{i\alpha(x-ct)}, \tag{1}$$

where $\varphi(y)$ is a complex amplitude, $\alpha = \alpha^* L^*$ the wave number, $c = c^*/U_\infty^*$ the wave velocity and $y = y^*/L^*$ the dimensionless independent variable.

With these assumptions the Orr-Sommerfeld equation becomes, see [11],

$$\frac{i}{\alpha R}\,(\varphi^{\mathrm{IV}} - 2\alpha^2\varphi'' + \alpha^4\varphi) + (U - c)\,(\varphi'' - \alpha^2\varphi) - U''\,\varphi = 0, \tag{2}$$

where $U(y)$ is the x-component of the velocity in the boundary layer (Blasius profile) and a prime denotes differentiation to y. The boundary conditions corresponding to (2) are

$$\begin{aligned} &\text{wall:} && \varphi(0) = \varphi'(0) = 0, \\ &\text{free stream} && \lim_{y\to\infty} \varphi(y) = \lim_{y\to\infty} \varphi'(y) = 0. \end{aligned} \tag{3}$$

The last condition excludes the continuous spectrum [10]. In this paper we are only interested in discrete eigenvalues with their eigenfunctions.

The homogeneous differential equation (2) with the homogeneous boundary conditions (3) defines an eigenvalue problem for the parameter c when α and R are given. In general, the eigenvalue c will be complex, $c = c_r + ic_i$.

If α is real, the perturbation (1) remains of constant amplitude in x-direction, but will increase with time if $c_i > 0$ and decrease if $c_i < 0$. This is called the temporal stability problem.

If the frequency $\omega = \alpha c$ is real, the perturbation will be of constant amplitude in time, but will decrease or increase in x-direction. This is the spatial stability problem, leading to complex values of both α and c.

A point of the neutral curve in the (R,α)-diagram is obtained if for real α also c becomes real. In that case the temporal and spatial problems lead to the same solution. The relation between the solutions of the two problems, if α and ω are not both real, is stated by Gaster [12].

We now consider the *asymptotic behaviour for $y \to \infty$* of the solutions of the Orr-Sommerfeld equation. Since for $y \to \infty$ the Blasius profile satisfies

$$U(y) \to 1, \quad U''(y) \to 0,$$

eq. (2) becomes

$$\frac{i}{\alpha R}(\varphi^{\mathrm{IV}} - 2\alpha^2\varphi'' + \alpha^4\varphi) + (1-c)(\varphi'' - \alpha^2\varphi) = 0, \tag{4}$$

which is a linear equation with constant coefficients. The four independent solutions of (2) are asymptotic to the solutions of (4):

$$\begin{aligned} &\varphi_j(y) \to \exp(\lambda_j y) \quad \text{as } y \to \infty, \quad j = 1,2,3,4. \\ &\lambda_{1,2} = \mp\gamma, \quad \lambda_{3,4} = \mp\alpha, \quad \gamma = \sqrt{\alpha^2 + i\alpha R(1-c)}, \quad Re\,\gamma \geq 0. \end{aligned} \tag{5}$$

Solutions 1 and 2 are the 'viscous' solutions, while 3 and 4 are 'inviscid' solutions. Only the solutions 1 and 3 satisfy the boundary conditions (3) at infinity. Hence, the eigenfunctions of the eigenvalue problem (2), (3) will be linear combinations of the solutions 1 and 3. The eigenvalue problem reduces to finding such values of c for which a linear combination of the solutions 1 and 3 satisfies the two boundary conditions (3) at the wall.

3. Numerical solution of the Orr-Sommerfeld equation

The numerical method to solve the Orr-Sommerfeld equation (2) is a direct integration method. It is a shooting procedure, starting at some large value $y = y_1$ and integrating toward $y = 0$. We consider the temporal stability problem, that is α will be taken real. Two of the four parameters R, α, c_r and c_i will be given fixed values, while the two others are given trial values. The final values of the two last parameters have to be determined by the shooting process.

The integration procedure is applied in order to obtain the solutions φ_1 and φ_3. Initial conditions at $y = y_1$ are

$$(\varphi_1, \varphi_1', \varphi_1'', \varphi_1''') = (1, -\gamma, \gamma^2, -\gamma^3)\, e^{-\gamma y_1},$$
$$(\varphi_3, \varphi_3', \varphi_3'', \varphi_3''') = (1, -\alpha, \alpha^2, -\alpha^3)\, e^{-\alpha y_1}. \tag{6}$$

The difficulty with this integration is that at smaller values of y the two solutions tend to become linearly dependent. This is exactly the reason that an orthogonalization process has been applied in [4] and [5].

To circumvent the difficulty of linear dependence of the solutions we use the *method of order reduction*. Assume that by integration we have obtained φ_1, which is the solution increasing fastest ($Re\,\gamma > \alpha$) in the direction of decreasing y. We then put

$$\varphi_3 = \psi\, \varphi_1 \tag{7}$$

and substitute this in eq. (2). This results in the following differential equation for ψ

$$\varphi_1 \psi^{\mathrm{IV}} + 4\varphi_1' \psi''' + (6\varphi_1'' - 2\alpha^2 \varphi_1)\psi'' - i\alpha R(U - c)\varphi_1 \psi''$$
$$+ 4(\varphi_1''' - \alpha^2 \varphi_1')\psi' - 2i\alpha R(U - c)\varphi_1' \psi' = 0. \tag{8}$$

It is seen that ψ = constant is a trivial solution of this equation, which corresponds to φ_1 being a solution of the original homogeneous equation (2).

In fact, (8) is a third order differential equation for ψ'. The initial conditions at $y = y_1$ for ψ' and its derivatives follow from (6) as

$$(\psi', \psi'', \psi''') = \{1, \gamma - \alpha, (\gamma - \alpha)^2\} e^{(\gamma - \alpha) y_1}.$$

Since γ is complex while α is real, it follows from (5) that asymptotically φ_1 is an oscillating and φ_3 a monotonous function. It turns out that the oscillations in φ_1 continue over the whole range of integration while oscillations are absent in φ_3. This means that the integration of both eq. (2) and eq. (8) produces oscillating functions and that the oscillations cancel when calculating φ_3. The oscillations can be avoided, which allows larger steps to be taken in the integration process for the same accuracy, by the following modification. Put

$$\varphi_1 = \eta\, \tilde{\varphi}_1, \tag{9}$$

where $\eta = e^{\mu y} = e^{-i\gamma_i y}$ and $\gamma = \gamma_r + i\gamma_i$. Then $\tilde{\varphi}_1$ does not have the strongly oscillating character of φ_1. The Orr-Sommerfeld equation written in terms of $\tilde{\varphi}$ is

$$\tilde{\varphi}^{\mathrm{IV}} + 4\mu\tilde{\varphi}''' + (6\mu^2 - 2\alpha^2)\tilde{\varphi}'' - i\alpha R(U - c)\tilde{\varphi}'' + 4\mu(\mu^2 - \alpha^2)\tilde{\varphi}' +$$
$$- 2i\mu\alpha R(U - c)\tilde{\varphi}' + [\mu^4 - 2\alpha^2\mu^2 + \alpha^4 - i\alpha R(U - c)(\mu^2 - \alpha^2) + \tag{10}$$
$$+ i\alpha R U'']\tilde{\varphi} = 0.$$

The initial condition at $y = y_1$ is

$$(\widetilde{\varphi}_1, \widetilde{\varphi}_1', \widetilde{\varphi}_1'', \widetilde{\varphi}_1''') = (1, -\gamma_r, \gamma_r^2, -\gamma_r^3)e^{-\gamma_r y_1}. \tag{11}$$

In order to eliminate the oscillation in ψ, we put

$$\psi' = \eta^{-1}\,\widetilde{\psi}'. \tag{12}$$

Substituting this and (9) in eq. (8), we obtain the following differential equation for $\widetilde{\psi}'$

$$\begin{aligned}&\widetilde{\varphi}_1\,\widetilde{\psi}^{\mathrm{IV}} + (4\widetilde{\varphi}_1' + \mu\widetilde{\varphi}_1)\,\widetilde{\psi}''' + \{6\widetilde{\varphi}_1'' + 4\mu\widetilde{\varphi}_1' + (\mu^2 - 2\alpha^2)\widetilde{\varphi}_1\}\,\widetilde{\psi}'' + \\ &+ \{4\widetilde{\varphi}_1''' + 6\mu\widetilde{\varphi}_1'' + 4(\mu^2 - \alpha^2)\widetilde{\varphi}_1' + \mu(\mu^2 - 2\alpha^2)\widetilde{\varphi}_1\}\,\widetilde{\psi}' + \\ &- i\alpha R(U-c)\{\widetilde{\varphi}_1\,\widetilde{\psi}'' + (2\widetilde{\varphi}_1' + \mu\widetilde{\varphi}_1)\,\widetilde{\psi}'\} = 0\end{aligned} \tag{13}$$

with the initial condition at $y = y_1$

$$(\widetilde{\psi}', \widetilde{\psi}'', \widetilde{\psi}''') = \{1, \gamma_r - \alpha, (\gamma_r - \alpha)^2\}e^{(\gamma_r - \alpha)y_1}.$$

In the integration of eqs. (10) and (13) we need the Blasius boundary layer profile $U(y)$. This can be obtained by integrating the Blasius equation

$$2f''' + ff'' = 0$$

with the boundary conditions $f(0) = f'(0) = 0$ and $\lim_{y\to\infty} f'(y) = 1$. In fact, this equation is integrated as an initial problem starting with

$$f(0) = f'(0) = 0. \quad f''(0) = 0.33205733622$$

in which case $f'(\infty)$ will become equal to 1. Then

$$U(y) = f'(y) \quad \text{and} \quad U''(y) = f'''(y).$$

The value of y_1 has been taken equal to 13, since then $U(y_1)$ and $U''(y_1)$ deviate less than 10^{-9} from the asymptotic values 1 and 0. Values of $U(y)$ and $U''(y)$ with a step length of 0.05 are retained in the memory. Intermediate values, required for the integration of eqs. (10) and (13), are obtained by Hermite interpolation, which gives an error everywhere smaller than 10^{-9}.

Integration of eqs. (10) and (13) corresponds to the integration of a system of 14 real first-order differential equations. The integration of this system has been performed by aid of the variable-stepsize, variable-order Adams-Bashforth-Moulton method DE, written by Shampine and Gordon [13]. Some results have been checked with the 7th order variable stepsize integrator written by Fehlberg* [14]. There was complete agreement within the required accuracy.

* In Table 2 of [14] $\beta_{\kappa\lambda}$ for $\kappa = 12$, $\lambda = 4$ should read $\frac{4496}{1025}$.

4. Determination of the eigenvalue

The method has first been tried out on the standard case $\tilde{R} = 998$, $\tilde{\alpha} = 0.308$ where $\tilde{R} = R\beta$ and $\tilde{\alpha} = \alpha\beta$ with β given in Sec. 2. The parameters c_r and c_i have then to be determined by shooting. The purpose of the shooting is to find such linear combination of the solutions φ_1 and φ_3 that the two boundary conditions (3) at the wall are satisfied. Using (7) this means that

$$\begin{aligned} &A\varphi_1(0) + B\psi(0)\varphi_1(0) = 0, \\ &A\varphi_1'(0) + B\psi'(0)\varphi_1(0) + B\psi(0)\varphi_1'(0) = 0. \end{aligned} \tag{14}$$

This system has only a significant solution for A/B if its determinant vanishes. Since $\varphi_1(0)$ never vanishes, as φ_1 is the solution that strongly increases with decreasing y, the determinant only vanishes if $\psi'(0) = 0$. Since $\eta = 1$ for $y = 0$ this leads, using (12), to the shooting criterion

$$\tilde{\psi}'(0) = 0. \tag{15}$$

A Newton-Raphson procedure is followed to determine such c_r and c_i that this criterion is satisfied. If $\tilde{\psi}'(0) \neq 0$ for trial values of c_r and c_i, corrections Δc_r and Δc_i are applied, resulting from

$$\tilde{\psi}'(0) + \Delta c_r \frac{\partial \tilde{\psi}'(0)}{\partial c_r} + \Delta c_i \frac{\partial \tilde{\psi}'(0)}{\partial c_i} = 0. \tag{16}$$

Since $\tilde{\psi}'(0)$ is a complex quantity, this gives two equations allowing the determination of both c_r and c_i. The derivatives of $\tilde{\psi}'(0)$ to c_r and c_i must be calculated by repeating the integration of eqs. (10) and (13) with values of c_r and c_i slightly varied from the original trial values.

After having determined improved values of c_r and c_i by aid of (16), the process is repeated. Once a fairly good approximation to the solution has been found, it may become superfluous to calculate new values of the derivatives at every step.

The result for the standard case is

$$\tilde{R} = 998, \quad \tilde{\alpha} = 0.308, \quad c_r = 0.36412129, \quad c_i = 0.00796250.$$

As a check, the adjoint to the Orr-Sommerfeld equation

$$\frac{i}{\alpha R}\left(\varphi^{\mathrm{IV}} - 2\alpha^2\varphi'' + \alpha^4\varphi\right) + (U - c)\left(\varphi'' - \alpha^2\varphi\right) + 2U'\varphi' = 0$$

has been dealt with in the same way as described in Sec. 3 for the original Orr-Sommerfeld equation. This led to exactly the same eigenvalue c.

In order to calculate the *neutral curve*, $\tilde{R}$ or $\tilde{\alpha}$ is given a fixed value, while c_i is taken equal to 0. By the Newton-Raphson procedure the other parameters $\tilde{\alpha}$ or $\tilde{R}$ and c_r are determined. Which of the two parameters $\tilde{R}$ or $\tilde{\alpha}$ is taken as fixed parameter depends on the direction of the tangent to the neutral curve. If the curve runs more in the direction of the $\tilde{\alpha}$-axis (small $\tilde{R}$) $\tilde{\alpha}$ is taken as fixed parameter, while in the remaining part $\tilde{R}$ is the fixed parameter.

Results for the neutral curve are given in Table 1 and Figs. 1 and 2.

TABLE 1.

Values of $\tilde{R}$, $\tilde{\alpha}$ and c_r for points on the neutral curve.

$\tilde{R}$	$\tilde{\alpha}$	c_r
10000.0000	0.06609819	0.18513238
3000.0000	0.10279770	0.24595028
2850.0000	0.10494983	0.24897783
2700.0000	0.10729132	0.25221520
2550.0000	0.10985237	0.25569052
2400.0000	0.11267043	0.25943773
2250.0000	0.11579262	0.26349829
2100.0000	0.11927919	0.26792358
1950.0000	0.12320866	0.27277817
1800.0000	0.12768569	0.27814480
1650.0000	0.13285351	0.28413168
1500.0000	0.13891453	0.29088406
1350.0000	0.14616646	0.29860301
1200.0000	0.15507038	0.30757824
1050.0000	0.16639135	0.31824962
987.9028	0.17207877	0.32331823
900.0000	0.18153267	0.33133691
734.2099	0.20649452	0.35025830
602.5375	0.24091027	0.37152551
536.5508	0.27532603	0.38757560
519.9570	0.30974178	0.39796245
584.5108	0.34415753	0.39896246
688.3151	0.35469717	0.39205508
750.0000	0.35615632	0.38771377
860.3938	0.35533729	0.38033325
900.0000	0.35444914	0.37784404
1050.0000	0.34972752	0.36917793
1200.0000	0.34404688	0.36158668
1350.0000	0.33815367	0.35488200
1500.0000	0.33235298	0.34890295
1650.0000	0.32677235	0.34352237
1800.0000	0.32146053	0.33864094
1950.0000	0.31642967	0.33418066
2100.0000	0.31167443	0.33007967
2250.0000	0.30718154	0.32628824
2400.0000	0.30293445	0.32276594
2550.0000	0.29891569	0.31947953
2700.0000	0.29510816	0.31640140
2850.0000	0.29149562	0.31350838
3000.0000	0.28806302	0.31078087
10000.0000	0.21217669	0.25191105

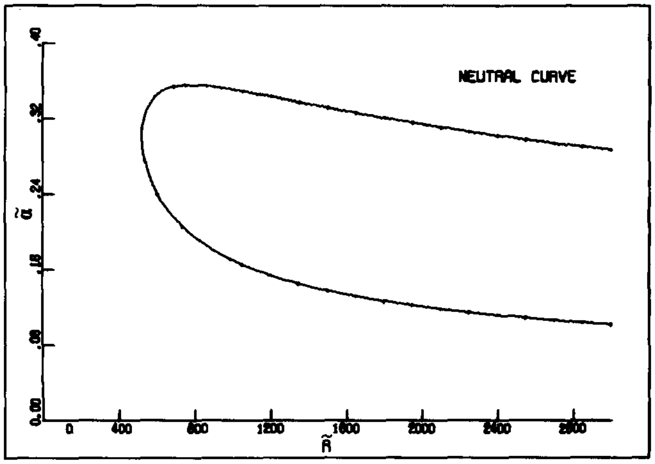

Figure 1. The neutral curve for the wave number $\tilde{\alpha}$ as a function of $\tilde{R}$ for the Blasius boundary-layer flow.

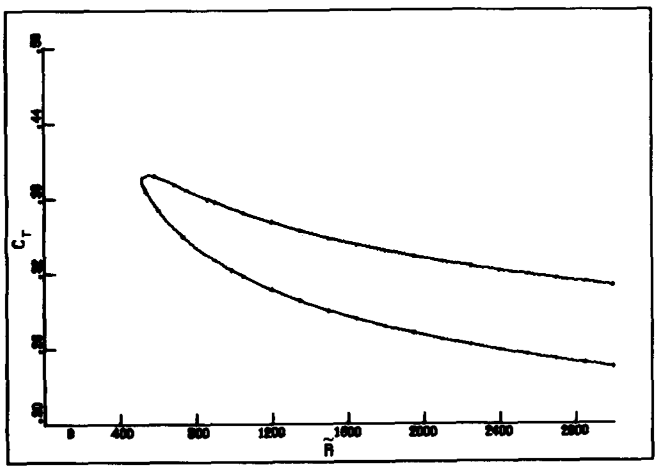

Figure 2. The neutral curve for the wave velocity c_r as a function of $\tilde{R}$ for the Blasius boundary-layer flow.

5. Determination of the eigenfunction

For the standard case and for some points of the neutral curve the eigenfunction has also been determined.

The eigenfunction is given by

$$\varphi(y) = A\varphi_1(y) + B\varphi_3(y).$$

The ratio A/B follows from the determinant of the system (14) to be equal to $-\psi$ (0). Using also (7), we find

$$\varphi(y) = \varphi_1(y)\{\psi(y) - \psi(0)\} = \varphi_1(y) \int_0^y \psi'(t)\,dt$$

or, with (9) and (12)

$$\varphi(y) = \eta(y)\tilde{\varphi}_1(y) \int_0^y \frac{\tilde{\psi}'(t)}{\eta(t)}\,dt. \tag{17}$$

Introduce $v(y) = \int_0^y \frac{\tilde{\psi}'(t)}{\eta(t)}\,dt$, then v satisfies the differential equation

$$v' = \frac{\tilde{\psi}'}{\eta} \text{ with } v(0) = 0.$$

However, integration of this differential equation simultaneously with eq. (13), that is in the direction of decreasing y, does not lead to correct values of v. The solution $v = C$ of the homogeneous equation makes that errors in $\tilde{\psi}'$ remain of the same magnitude during the integration procedure. Since $\tilde{\psi}'$ is a rapidly decreasing function toward smaller values of y, this spoils the accuracy. The remedy is to divide the integration interval into N small subintervals (y_{i-1}, y_i), $i = 1, 2, \ldots, N$ and calculate for each subinterval

$$s_i = v(y_i) - v(y_{i-1}),$$

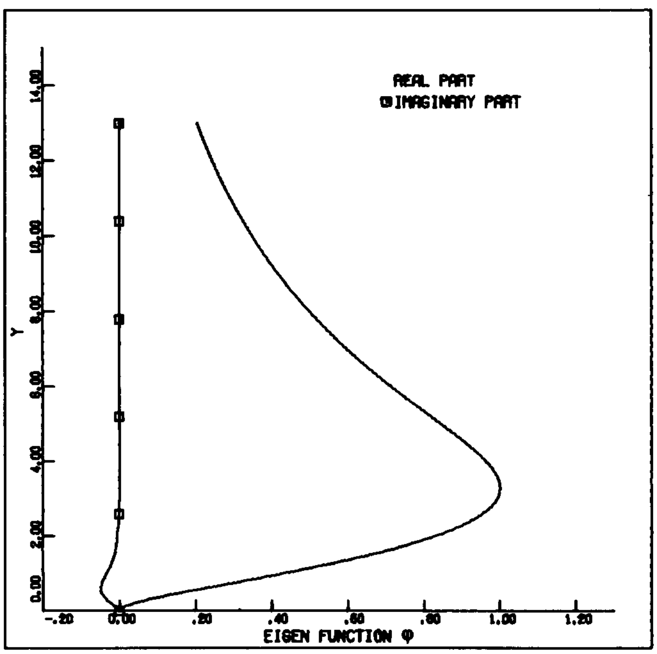

Figure 3. The eigenfunction $\varphi(y)$ for the case $\tilde{R} = 998$ and $\tilde{\alpha} = 0.308$.

which is done simultaneously with the integration of eqs. (10) and (13). The results for s_i are stored in the memory and after having calculated them all, we find

$$v(y_i) = \sum_{k=1}^{i} s_k, \quad i = 1, 2, \ldots, N.$$

Finally, from eq. (17) the eigenfunction becomes

$$\varphi(y) = \eta(y)\,\tilde{\varphi}_1(y)\,v(y).$$

The eigenfunction for the standard case ($\tilde{R} = 998, \tilde{\alpha} = 0.308$) is given in Fig. 3. Eigenfunctions for points of the neutral curve are qualitatively identical, the main difference being in the asymptotic approach to 0 for $y \to \infty$, which is slower for smaller $\tilde{\alpha}$.

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